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Metric Möbius geometry and a characterization of spheres

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Abstract. We obtain a Möbius characterization of the n-dimensional spheres S^n endowed with the chordal metric d_0. We show that every compact extended Ptolemy metric space with the property that every three points are contained in a circle is Möbius equivalent to (S^n, d_0) for some $n \geq 1$.

1. Introduction

In this paper, we characterize the spheres in the context of metric Möbius geometry by the existence of many circles. Therefore we define the notion of a Möbius structure $\mathcal{M}$ on a set X. Roughly speaking, a Möbius structure is an equivalence class of (extended) metrics on X, where two metrics are equivalent, if they define the same crossratio. A pair $(X, \mathcal{M})$ is then called a Möbius space. The classical example of a Möbius space is $(S^n, [d_0])$, where S^n is the sphere and the Möbius structure $\mathcal{M} = [d_0]$ is given by the equivalence class of the chordal metric d_0 on S^n. Via the stereographic projection this space is Möbius equivalent to $(\mathbb{E}^n \cup \{\infty\}, [d])$, where d is the (extended) euclidean metric on $\mathbb{E}^n \cup \{\infty\}$.

We say that a Möbius structure $\mathcal{M}$ on X is Ptolemy, if for any $d \in \mathcal{M}$ and any four points points $x_1, x_2, x_3, x_4 \in X$,

$$d(x_1, x_3)\, d(x_2, x_4) \;\leq\; d(x_1, x_2)\, d(x_3, x_4) \;+\; d(x_1, x_4)\, d(x_2, x_3). \tag{1}$$

Due to Theorem 2.1 this is a very natural condition for a Möbius structure. The classical structure $\mathcal{M} = [d_0]$ on S^n is Ptolemy.

For a Möbius space $(X, \mathcal{M})$ one can naturally define the notion of a circle $\sigma \subset X$. In the classical example every three points are contained in a circle. Our main result is a characterization by that property.

Theorem 1.1. *Let $(X, \mathcal{M})$ be a compact Ptolemy Möbius space which contains at least three points. If any three points in X lie on a circle, then $(X, \mathcal{M})$ is Möbius equivalent to $(S^n, [d_0])$ for some $n \in \mathbb{N}$.*

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The proof relies on a result of Hitzelberger and Lytchak on spaces with many affine functions and on a characterization of normed Ptolemy spaces by Schoenberg.

Our motivation to investigate Ptolemy Möbius structures comes from the study of CAT(-1)-spaces (see [5]): Let Y be a CAT(-1)-space and $X = \partial_\infty Y$ its ideal boundary, then X carries in a natural way a Ptolemy Möbius structure $\mathcal{M} = [\rho_o] = [\rho_\omega]$, where ρ_o (resp. ρ_ω) is the Bourdon, [1] (resp. Hamenstädt, [7]) metric with respect to a point $o \in Y$ (resp. $\omega \in X$). The circles in X correspond uniquely to the boundaries $\partial_\infty W$ of isometrically embedded subsets $W \subset Y$ isometric to the hyperbolic plane H^2.

Since Ptolemy Möbius structures arise naturally in the context of negatively curved spaces, we think that they should be studied as own subject in detail and the present paper is a first step to study the rigidity properties of these structures. For further investigations compare e.g. [3].

We emphasize that one can generalize Theorem 1.1 in order to obtain an analogue characterization of chordal hemispheres. Moreover we can classify all Ptolemy circles as well as provide examples of higher dimensional Möbius spheres which are not Möbius equivalent to the chordal spheres. Since, however, these considerations are technically much more involved, details will be given elsewhere.

The paper is organized as follows: in Sect. 2 we give a short introduction to metric Möbius geometry and Ptolemy structures. In Sect. 3 we state the result on affine functions, which we will need in the proof which is carried out in Sect. 4.

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2. Metric Möbius geometry

2.1. Möbius structure

Let X be a set which contains at least two points. An *extended metric* on X is a map $d : X \times X \to [0, \infty]$, such that there exists a set $\Omega(d) \subset X$ with cardinality $\#\Omega(d) \in \{0, 1\}$, such that d restricted to the set $X \setminus \Omega(d)$ is a metric (taking only values in $[0, \infty)$) and such that $d(x, \omega) = \infty$ for all $x \in X \setminus \Omega(d)$, $\omega \in \Omega(d)$. Furthermore $d(\omega, \omega) = 0$.

If $\Omega(d)$ is not empty, we sometimes denote $\omega \in \Omega(d)$ simply as ∞ and call it the (infinitely) remote point of (X, d). We often write also $\{\omega\}$ for the set $\Omega(d)$ and X_ω for the set $X \setminus \{\omega\}$.

The topology considered on (X, d) is the topology with the basis consisting of all open distance balls $B_r(x)$ around points in $x \in X_\omega$ and the complements D^C of all closed distance balls $D = \overline{B}_r(x)$.

We call an extended metric space *complete*, if first every Cauchy sequence in X_ω converges and secondly if the infinitely remote point ω exists in case that X_ω is unbounded. For example the real line $(\mathbb{R}, d)$, with its standard metric is *not* complete (as an extended metric space), while $(\mathbb{R} \cup \{\infty\}, d)$ is complete.

We say that a quadruple $(x, y, z, w) \in X^4$ is *admissible*, if no entry occurs three or four times in the quadruple. We denote with $Q \subset X^4$ the set of admissible quadruples. We define the *cross ratio triple* as the map $\mathrm{crt} : Q \to \Sigma \subset \mathbb{R}P^2$ which maps admissible quadruples to points in the real projective plane defined by

$$\mathrm{crt}(x, y, z, w) = (d(x, y)d(z, w) : d(x, z)d(y, w) : d(x, w)d(y, z)),$$

here Σ is the subset of points $(a : b : c) \in \mathbb{R}P^2$, where all entries a, b, c are nonnegative or all entries are non-positive. Note that Σ can be identified with the standard 2-simplex, $\{(a, b, c) \mid a, b, c \geq 0,\ a + b + c = 1\}$.

We use the standard conventions for the calculation with ∞. If ∞ occurs once in Q, say $w = \infty$, then $\operatorname{crt}(x, y, z, \infty) = (d(x, y) : d(x, z) : d(y, z))$. If ∞ occurs twice , say $z = w = \infty$ then $\operatorname{crt}(x, y, \infty, \infty) = (0 : 1 : 1)$.

Similar as for the classical cross ratio there are six possible definitions by permuting the entries. The particular choice is not essential and we choose the above one.

It is not difficult to check that $\operatorname{crt} : Q \to \Sigma$ is continuous, where Q and Σ carry the obvious topologies induced by X and $\mathbb{R}P^2$. Thus, if $(x_i, y_i, z_i, w_i) \in Q$ for $i \in \mathbb{N}$ and assume $x_i \to x, \ldots, w_i \to w$, where $(x, y, z, w) \in Q$ then $\operatorname{crt}(x_i, y_i, z_i, w_i) \to \operatorname{crt}(x, y, z, w)$.

A map $f : X \to Y$ between two extended metric spaces is called *Möbius*, if f is injective and for all admissible quadruples (x, y, z, w) of X,

$$\operatorname{crt}(f(x), f(y), f(z), f(w)) = \operatorname{crt}(x, y, z, w).$$

Möbius maps are continuous.

Two extended metric spaces (X, d) and (Y, d') are *Möbius equivalent*, if there exists a bijective Möbius map $f : X \to Y$. In this case also f^{-1} is a Möbius map and f is in particular a homeomorphism.

We say that two extended metrics d and d' on a set X are *Möbius equivalent*, if the identity map $\mathrm{id} : (X, d) \to (X, d')$ is a Möbius map. Möbius equivalent metrics define the same topology on X.

A *Möbius structure* on a set X is a nonempty set $\mathcal{M}$ of extended metrics on X, which are pairwise Möbius equivalent and which is maximal with respect to that property.

A Möbius structure defines a topology on X. In general two metrics in $\mathcal{M}$ can look very different. However if two metrics have the same remote point at infinity, then they are homothetic. Since this result is crucial for our considerations, we state it as a Lemma.

Lemma 2.1. *Let $\mathcal{M}$ be a Möbius structure on a set X, and let $d, d' \in \mathcal{M}$, such that $\omega \in X$ is the remote point of d and of d'. Then there exists $\lambda > 0$, such that $d'(x, y) = \lambda d(x, y)$ for all $x, y \in X$.*

Proof. Since otherwise the result is trivial, we can assume that there are distinct points $x, y \in X_\omega$. Choose $\lambda > 0$ such that $d'(x, y) = \lambda d(x, y)$. If $z \in X_\omega$, then $\operatorname{crt}(x, y, z, \omega)$ is the same in the metric d and d', hence$(d'(x, y) : d'(x, z) : d'(y, z)) = (d(x, y) : d(x, z) : d(y, z))$. Since $d'(x, y) = \lambda d(x, y)$ we therefore obtain $d'(x, z) = \lambda d(x, z)$ and $d'(y, z) = \lambda d(y, x)$. $\square$

2.2. *Ptolemy Möbius structures*

An extended metric space (X, d) is called a *Ptolemy space*, if for all quadruples of points $\{x, y, z, w\} \in X^4$ the *Ptolemy inequality* holds

$$d(x, z)\, d(y, w) \ \leq\ d(x, y)\, d(z, w) \ +\ d(x, w)\, d(y, z)$$

We can reformulate this condition in terms of the cross ratio triple. Let $\Delta \subset \Sigma$ be the set of points $(a : b : c) \in \Sigma$, such that the entries a, b, c satisfy the triangle inequality, i.e. all inequalities $a \le b+c, b \le a+c, c \le a+b$ hold. This is obviously well defined. If we identify $\Sigma \subset \mathbb{R}P^2$ with the standard 2-simplex, i.e. the convex hull of the unit vectors e_1, e_2, e_3, then Δ is the convex subset spanned by $(0, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2}, 0)$. Note that also Δ is homeomorphic to a 2-simplex and $\partial\Delta$ is homeomorphic to S^1. Then an extended space is Ptolemy, if $\mathrm{crt}(x, y, z, w) \in \Delta$ for all allowed quadruples Q.

This description shows that the Ptolemy property is Möbius invariant: if (X, d) is Ptolemy and if (X', d') is Möbius equivalent to (X, d), then (X', d') is Ptolemy as well. In particular, if $\mathcal{M}$ is a Möbius structure on X, and $d, d' \in \mathcal{M}$, then (X, d) is Ptolemy if and only if (X, d') is Ptolemy. Thus the Ptolemy property is a property of the Möbius structure $\mathcal{M}$.

The importance of the Ptolemy property comes from the following fact.

Theorem 2.1. *A Möbius structure $\mathcal{M}$ on a set X is Ptolemy, if and only if for all $z \in X$ there exists $d_z \in \mathcal{M}$ with $\Omega(d_z) = \{z\}$.*

Proof. Assume that $\mathcal{M}$ is Ptolemy and that $z \in X$. Choose some $d \in \mathcal{M}$. If $z \in \Omega(d)$, we have our desired metric. If not we define $d_z : X \times X \to [0, \infty]$ by

$$\begin{array}{lcll} d_z(x, y) & = & \frac{d(x,y)}{d(z,x)d(z,y)} & \text{for} \quad x, y \in X \setminus (\Omega(d) \cup \{z\}), \\ d_z(x, \omega) & = & \frac{1}{d(z,x)} & \text{for} \quad x \in X \setminus \Omega(d), \omega \in \Omega(d) \\ d_z(z, x) & = & \infty & \text{for} \quad x \in X \setminus \{z\} \end{array}$$

Since for $x, y, w \in X \setminus \{z\}$

$$\begin{aligned} (d_z(x, y) : d_z(y, w) : d_z(x, w)) \quad = \\ (d(x, y)\, d(z, w) : d(x, z)\, d(y, w) : d(x, w)\, d(y, z)) \in \Delta \end{aligned}$$

we see that d_z satisfies the triangle inequality and hence $d_z \in \mathcal{M}$.

If on the other hand for every $z \in X$ there is a metric $d_z \in \mathcal{M}$ with $\Omega(d_z) = \{z\}$, then for all $x, y, w \in X \setminus \{z\}$ and all $d \in \mathcal{M}$

$$\begin{aligned} &(d(x, y)\, d(z, w) : d(x, z)\, d(y, w) : d(x, w)\, d(y, z)) \\ &\quad = (d_z(x, y)\, d_z(z, w) : d_z(x, z)\, d_z(y, w) : d_z(x, w)\, d_z(y, z)) \\ &\quad = (d_z(x, y) : d_z(y, w) : d_z(x, w)) \in \Delta \end{aligned}$$

which implies the Ptolemy inequality. □

Remark 2.2. The proof shows that a metric space (X, d) is Ptolemy if and only if the function d_z as defined above is a metric on $X \setminus \{z\}$ for all $z \in X$.

For other results on Ptolemy spaces compare [2] ,[4], [6], [9].

2.3. *Schoenberg's theorem*

A main ingredient of our proof is the following theorem due to Schoenberg.

Theorem 2.3. *([10]) A normed vector space* $(V, ||\cdot||)$ *is a Ptolemy metric space if and only if it is Euclidean.*

This, together with the fact that the Ptolemy condition is invariant under scaling, yields the

Corollary 2.4. *An open subset of a normed vector space* $(V, ||\cdot||)$ *is a Ptolemy space, if and only if* $(V, ||\cdot||)$ *is Euclidean.*

2.4. *Circles in ptolemy Möbius spaces*

A circle in a Ptolemy space (X, d) is a subset $\sigma \subset X$ homeomorphic to S^1 such that for distinct points $x, y, z, w \in \sigma$ (in this order)

$$d(x,z)d(y,w) = d(x,y)d(z,w) + d(x,w)d(y,z) \tag{2}$$

Here the phrase "in this order" means that y and w are in different components of $\sigma \setminus \{x, z\}$. We recall that the classical Ptolemy theorem states, that four points x, y, z, w of the euclidean plane lie on a circle (in this order), if and only if their distances satisfy the Ptolemy equality (2). One can reformulate this via the crossratio triple. A subset σ homeomorphic to S^1 is a circle, if and only if for all admissible quadruples (x, y, z, w) of points in σ we have $\mathrm{crt}(x, y, z, w) \in \partial\Delta$. This shows that a circle is Möbius invariantly defined and hence a concept of the Möbius structure. Let σ be a circle and let $\omega \in \sigma$ and consider $\sigma_\omega = \sigma \setminus \{\omega\}$ in a metric with remote point ω, then $\mathrm{crt}(x, y, z, \omega) \in \partial\Delta$ says that for $x, y, z \in \sigma_\omega$ (in this order) $d(x, y)+d(y, z) = d(x, z)$, i.e. it implies that σ_ω is a geodesic, actually a complete geodesic isometric to $\mathbb{R}$.

3. Affine functions

In this short section we review the theorem of Hitzelberger and Lytchak, which we will use in our proof.

Let X be a geodesic metric space. For $x, y \in X$ we denote by $m(x, y) = \{z \in X \mid d(x, z) = d(z, y) = \frac{1}{2}d(x, y)\}$ the set of midpoints of x and y. A map $f : X \to Y$ between two geodesic metric spaces is called *affine*, if for all $x, y \in X$, we have $f(m(x, y)) \subset m(f(x), f(y))$, i.e. midpoints are mapped to midpoints. Thus a map is affine if and only if it maps geodesics parameterized proportionally to arclength into geodesics parameterized proportionally to arclength. An affine map $f : X \to \mathbb{R}$ is called an affine function.

Definition 3.1. Let X be a geodesic metric space. We say that affine functions on X *separate points*, if for every $x, y \in X$, $x \neq y$, there exists an affine function $f : X \longrightarrow \mathbb{R}$ with $f(x) \neq f(y)$.

The following rigidity theorem, which is due to Hitzelberger and Lytchak, is a main tool in our argument.

Theorem 3.2. *([8]) Let X be a geodesic metric space. If the affine functions on X separate points, then X is isometric to a convex subset of a (strictly convex) normed vector space.*

4. Proof of the main result

In this section we prove Theorem 1.1.

Let $(X, \mathcal{M})$ be a compact Ptolemy Möbius structure such that each triple of points $x, y, z \in X$ is contained in a circle. We choose an arbitrary point $\omega \in X$ and consider the metric space (X_ω, d), where $X_\omega = X \setminus \{\omega\}$ and $d = d_{\omega|X_\omega}$. This is a geodesic space. Indeed, for any $x, y \in X_\omega$ a circle $\sigma \subset X$, with $x, y, \omega \in \sigma$ induces a geodesic line $\sigma \setminus \{\omega\}$ in X_ω, which contains x and y.

Let $c : [0, \infty) \to X_\omega$ be a geodesic ray parameterized by arclength. As usual we define the *Busemann function* $b_c(x) = \lim_{t\to\infty}(d(x, c(t)) - t)$.

Proposition 4.1. *Busemann functions* $b : X_\omega \to \mathbb{R}$ *are affine.*

Proof. Let $c : [0, \infty) \longrightarrow X_\omega$ be an arbitrary geodesic ray in (X_ω, d), and let $b = b_c$ be its Busemann function.

We first show that b is convex. Consider points $z, x, m, y \in X_\omega$, where m is a midpoint of x and y, i.e. $d(x, m) = \frac{1}{2}d(x, y) = d(m, y)$. Applying the Ptolemy inequality to the quadruple x, m, y, z, we obtain that

$$d(z, m) \leq \frac{1}{2}(d(z, x) + d(z, y))$$

which implies the convexity of the distance function $d(z, \cdot)$. Since Busemann functions are limits of distance functions they are also convex. Thus we have

$$b(m) \leq \frac{1}{2}(b(x) + b(y)).$$

To prove that b is affine, we need to prove the opposite inequality, i.e. we have to prove:

$$b(m) \geq \frac{1}{2}(b(x) + b(y)) \tag{3}$$

Consider $w_i = c(i)$ for $i \to \infty$. By assumption there exists a circle σ_i containing x, y, w_i.

We consider the subsegment of this circle which contains x and w_i as boundary points and y as interior point. On this segment we consider the orientation that $x < y < w_i$ and we choose for i large enough $u_i \in \sigma_i$ with $x < y < u_i < w_i$, such that $d(y, u_i) = a$, where $a = \frac{1}{2}d(x, y)$, and hence $d(x, u_i) \leq 3a$. By the Ptolemy equality on the circle σ_i applied to the quadruple x, y, u_i, w_i, we have

$$3a\, d(y, w_i) \geq d(x, u_i)\, d(y, w_i) = 2a\, d(u_i, w_i) + a\, d(x, w_i)$$

and hence

$$d(y, w_i) \geq \frac{2}{3}d(u_i, w_i) + \frac{1}{3}d(x, w_i). \tag{4}$$

Since X_ω is locally compact, some subsequence converges, thus we can assume $u_i \to u$ and x, y, u lie on a geodesic. We do not assume that m lies on this line.

Since $u_i \to u$, the Inequality (4) implies in the limit for the Busemann function

$$b(y) \geq \frac{2}{3}b(u) + \frac{1}{3}b(x). \tag{5}$$

Since x, y, u are on a geodesic and $d(y, u) = a$, we have $d(x, u) = 3a$. By triangle inequalities this implies $d(m, u) = 2a$, and hence y is a midpoint of m and u. The convexity of b implies

$$b(y) \leq \frac{1}{2}(b(m) + b(u)). \tag{6}$$

Now an easy computation shows that Inequalities (5) and (6) imply the desired estimate (3).

We are now ready to complete the proof of Theorem 1.1. Let $x, y \in X_\omega$ be distinct points, and σ be the circle through x, y, ω. Then the Busemann function corresponding to the line $\sigma \setminus \{\omega\}$ has different values at x and y. Thus the affine functions separate points. From Theorem 3.2, we deduce (X_ω, d) is a convex subset of a (strictly convex) normed vector space.

Since any two points lie on a geodesic line, one easily sees that all geodesic segments extend to complete lines. Thus we see that (X_ω, d) is itself isometric to a normed vector space. Since it also is a Ptolemy space, Schoenberg's Theorem, Theorem 2.3, implies that it is a Euclidean space. Since $(X, \mathcal{M})$ is compact, (X_ω, d) is locally compact and hence isometric to $\mathbb{E}^n$ for some $n \in \mathbb{N}$. Thus (X, d_ω) is isometric to $(\mathbb{E}^n \cup \{\infty\}, d)$, where d is the extended euclidean metric and hence $(X, \mathcal{M})$ Möbius equivalent to the classical Möbius structure on the sphere. □

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